SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

CANDIDATE NUMBER

2023 Trial Examination

Form VI Mathematics Extension 1

Thursday 10th August 2023
12:50pm

General Instructions

- Reading time — 10 minutes
- Working time — 2 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 70

Section I (10 marks) Questions 1 – 10

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (60 marks) Questions 11 – 14

- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- Your name and master should only be written on this page.
- Write your candidate number on this page, on each booklet and on the multiple choice sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.
- Place everything inside this question booklet.

Checklist

- Reference sheet
- Multiple-choice answer sheet
- 4 booklets per boy
- Candidature: 131 pupils

Writer: GMC

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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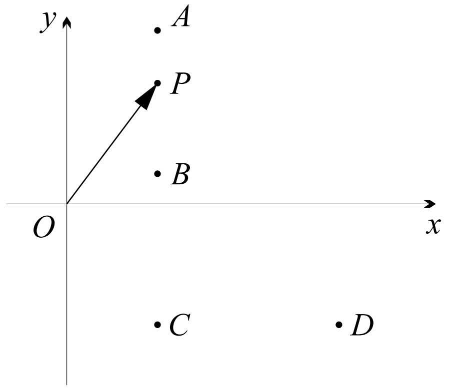
1. The diagram shows the points A, B, C, D and P and the vector $\overrightarrow{OP}$.



The projection of which of the vectors below onto $\overrightarrow{OP}$ has the greatest magnitude?

(A) $\overrightarrow{OA}$

(B) $\overrightarrow{OB}$

(C) $\overrightarrow{OC}$

(D) $\overrightarrow{OD}$

2. A car dealer buys 139 cars, each of which is painted one of four colours: red, green, blue or white. Consider the statement: "At least X of the cars must be the same colour". By the pigeonhole principle, what is the largest value of X for which this statement is true?

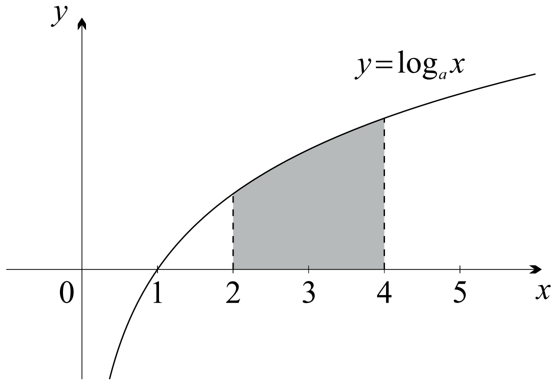
(A) 34

(B) 35

(C) 36

(D) 37

3. Consider the area between the curve $y = \log_a x$, the x-axis, $x = 2$ and $x = 4$ as shown in the diagram.



What is the area approximated by one application of the trapezoidal rule?

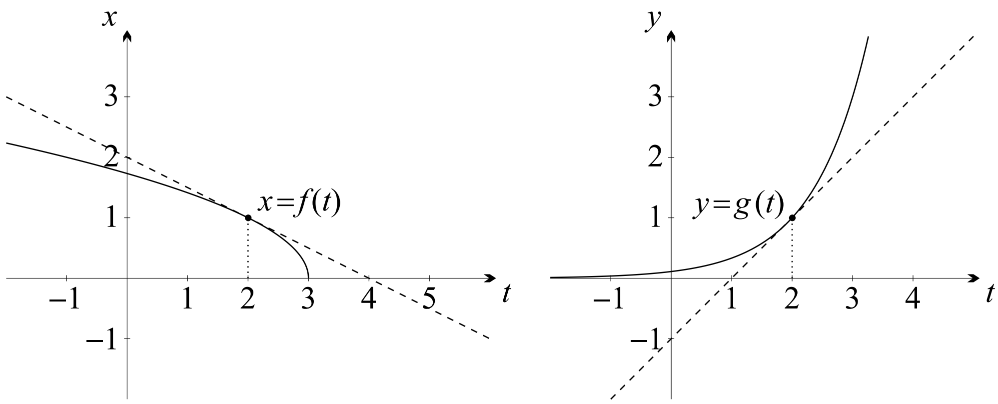
(A) $3\log_a 2$

(B) $\log_a 6$

(C) $\dfrac{\log_a 2 + \log_a 4}{2}$

(D) $\dfrac{3a}{2}$

4. A curve $\mathcal{C}$ is defined parametrically as $x = f(t)$ and $y = g(t)$, the graphs of which are shown in the diagram.



What is the gradient of the tangent to the curve $\mathcal{C}$ when $t = 2$?

(A) 2

(B) $\frac{1}{2}$

(C) $-\frac{1}{2}$

(D) -2

5. If $f(x) = 2\cos\dfrac{x}{2}$, what is the natural domain of $y = f^{-1}(x)$?

   (A) $[0, 2]$

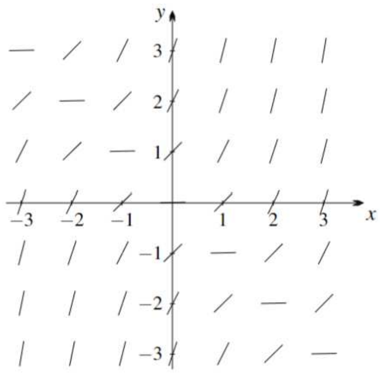
   (B) $[-2, 2]$

   (C) $[-\pi, \pi]$

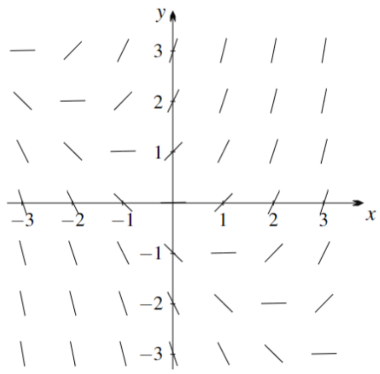
   (D) $[0, 2\pi]$

6. Which of the following is the slope field of $\dfrac{dy}{dx} = |x + y|$?

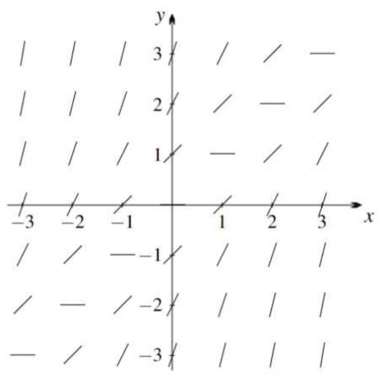
   (A)



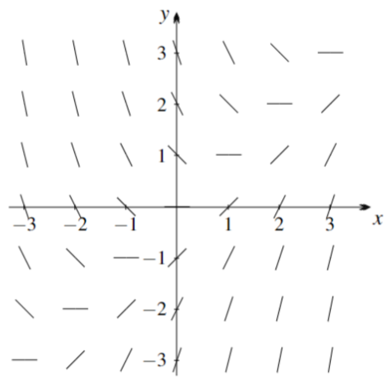
   (C)



   (B)



   (D)



7. A year group of 200 students consists of 100 girls and 100 boys. In how many ways can 20 prefects be chosen from the year group if the prefect group has an equal number of girls and boys?

   (A) ${}^{200}\mathrm{C}_{10}\ {}^{190}\mathrm{C}_{10}$

   (B) ${}^{100}\mathrm{C}_{10}\ {}^{100}\mathrm{C}_{10}$

   (C) $\dfrac{{}^{200}\mathrm{C}_{20}}{{}^{100}\mathrm{C}_{10}\ {}^{100}\mathrm{C}_{10}}$

   (D) $\dfrac{{}^{200}\mathrm{C}_{20}}{2!}$

8. A museum sells adult and concession tickets. Let the components of $\underset{\sim}{u}$ be the number of adult tickets and the number of concession tickets sold in a day, and the components of $\underset{\sim}{v}$ be the price of an adult ticket and the price of a concession ticket. What does the dot product $\underset{\sim}{u} \cdot \underset{\sim}{v}$ represent?

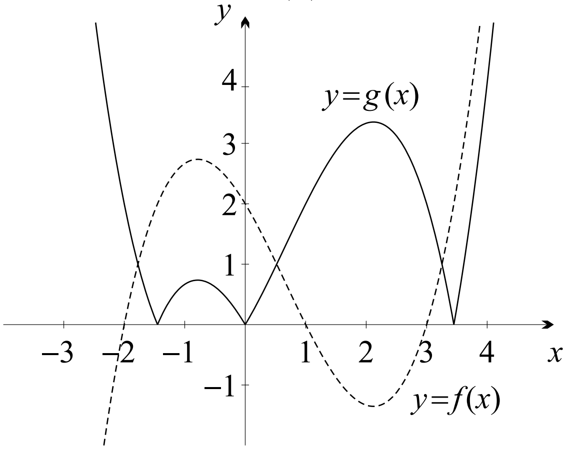
    (A) The average ticket price

    (B) The total number of tickets sold on that day

    (C) The total value of the tickets sold on that day

    (D) The cosine of the angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$

9. The graph of $y = f(x)$ is shown with a dashed line in the diagram. The graph of $y = g(x)$ is a transformation of $y = f(x)$, and is shown with a solid line.



What is the equation of $y = g(x)$?

    (A) $g(x) = |f(x) + 1|$

    (B) $g(x) = f(|x - 1|)$

    (C) $g(x) = |f(x) - 2|$

    (D) $g(x) = |f(|x| - 2)|$

10. Consider $y = f(x)$ on the the domain $(0, \infty)$ where $f'(x) > 0$. Let $g(x) = f\left(\dfrac{1}{x}\right)$.

    Given $y = f(x)$ and $y = g(x)$ intersect at $x = a$, which of the following is NOT ALWAYS TRUE?

    (A) $g'(a) < 0$

    (B) $g'(a) = -f'(a)$

    (C) $g'(a) = -f'\left(\dfrac{1}{a}\right)$

    (D) $g'(a) = \dfrac{-1}{f'(a)}$

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet. **Marks**

(a) Show $x + 1$ is a factor of $P(x) = x^{17} - 2x - 1$. 1

(b) Consider the differential equation $\dfrac{dy}{dx} = y(1-x)$. Calculate the gradient of the solution curve passing through the point $(3, 2)$. 1

(c) Which boy's name has more unique arrangements of letters: MILO or SAVVAS? Show your working. 2

(d) Evaluate $\displaystyle\int \cos^2 2x \, dx$. 2

(e) Solve $\dfrac{x}{x+1} > 3$. 3

(f) Use mathematical induction to prove that $5n^2 - 3n$ is divisible by 2 for all positive integers n. 3

(g) The position in metres of a particle at time t seconds is given by $\underset{\sim}{x} = \begin{bmatrix} 3\cos t \\ 4\sin t \end{bmatrix}$. 3

By finding the velocity $\underset{\sim}{v}$ of the particle, calculate the particle's minimum and maximum speed.

QUESTION TWELVE (14 marks) Start a new answer booklet. **Marks**

(a) Calculate $\int_0^1 x(1-x)^7\,dx$ using the substitution $u = 1 - x$. 3

(b) The current world population is approximately 8 billion people. Assuming the 'carrying capacity' of the earth is 25 billion people, Anna models the population P after t years using the differential equation:

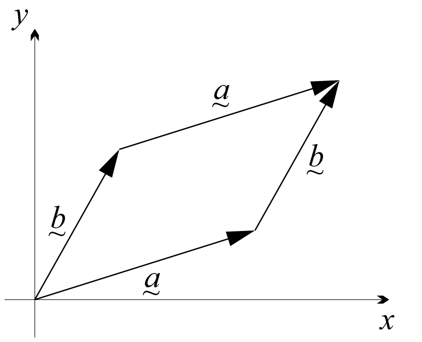
$$\frac{dP}{dt} = k(25 - P)$$

(i) Show that $P = 25 - 17e^{-kt}$ satisfies the differential equation above. 1

(ii) Find k if the world population after 1 year is 8.072 billion people. Write your answer correct to 3 significant figures. 2

(iii) After how many years does Anna's model say the world population will reach 20 billion people? Round your answer to the nearest whole year. 1

(c) Consider the parallelogram formed by the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as shown in the diagram.

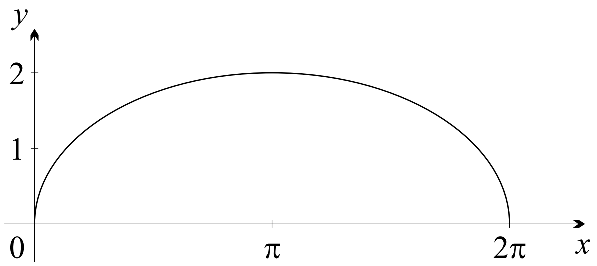


(i) Write down vectors representing the diagonals of the parallelogram in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. 1

(ii) Using vectors, show that the diagonals of the parallelogram are perpendicular when $|\underset{\sim}{a}| = |\underset{\sim}{b}|$. 2

(d) The curve below, called a cycloid, is defined parametrically as:

$$x = t - \sin t, \quad y = 1 - \cos t, \quad t \in [0, 2\pi]$$



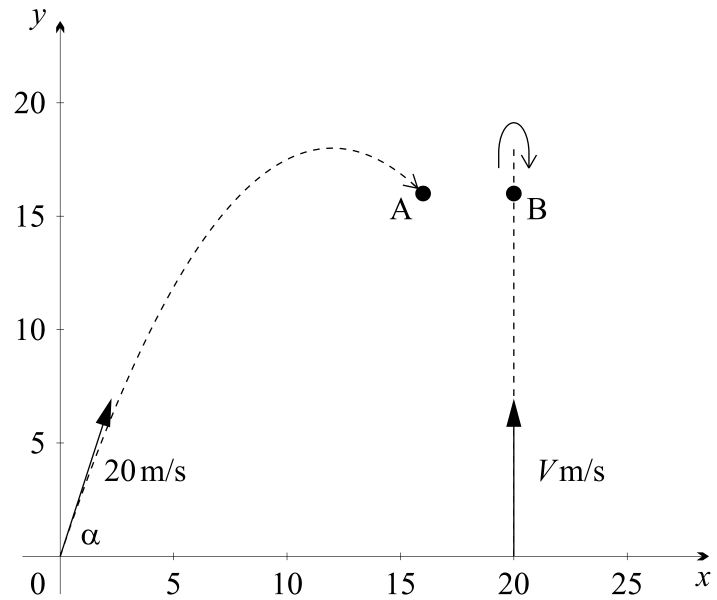
(i) Show that $\dfrac{dy}{dx} = \dfrac{\sin t}{1 - \cos t}$. 1

(ii) Hence find the value of t where the gradient of the tangent to the curve is $\dfrac{1}{\sqrt{3}}$. 3

QUESTION THIRTEEN (16 marks) Start a new answer booklet. **Marks**

(a) Expand and simplify $\left(x - \dfrac{2}{x}\right)^4$. $\boxed{2}$

(b) Two objects are launched at the same time from horizontal ground 20 m apart. Object A is projected from the origin at 20 m/s at an angle α degrees towards Object B and Object B is projected vertically at V m/s. The objects collide on their downward journey 10 m above the ground.



The positions $\underset{\sim}{r}_A$ and $\underset{\sim}{r}_B$ of Object A and Object B respectively after t seconds are given by the vector equations below (you do NOT need to prove these).

$$\underset{\sim}{r}_A = \begin{bmatrix} 20t\cos\alpha \\ -5t^2 + 20t\sin\alpha \end{bmatrix} \text{ and } \underset{\sim}{r}_B = \begin{bmatrix} 20 \\ -5t^2 + Vt \end{bmatrix}$$

(i) Given that $45^\circ < \alpha < 90^\circ$, show that $\alpha \approx 72^\circ$. $\boxed{3}$

(ii) Hence find V. Write your answer correct to 2 decimal places. $\boxed{2}$

(c) (i) Show that $\dfrac{2}{y(2-y)} = \dfrac{1}{y} + \dfrac{1}{2-y}$. $\boxed{1}$

(ii) Let $y = f(x)$ be the solution to $\dfrac{dy}{dx} = y(2-y)$ where $y(0) = 1$. $\boxed{3}$

Solve the differential equation to show that $f(x) = \dfrac{2}{1 + e^{-2x}}$.

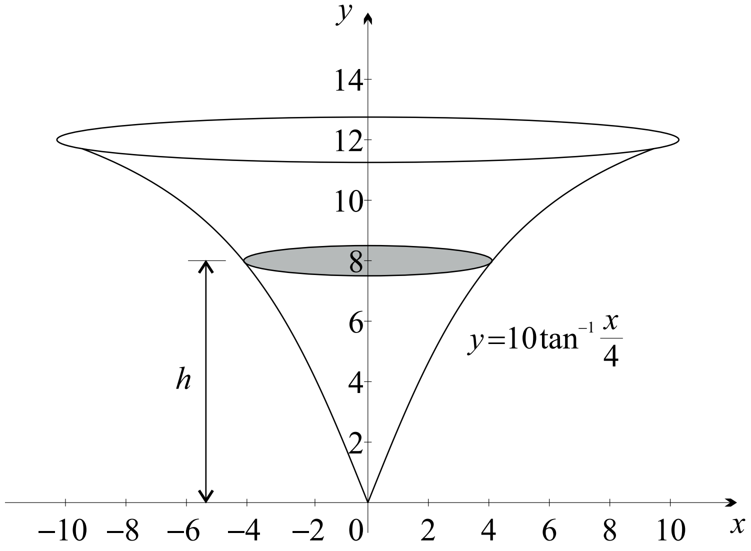
(iii) Show that $y = f(x)$ has point symmetry around $(0, 1)$ by showing that $g(x) = f(x) - 1$ is an odd function. $\boxed{2}$

(iv) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. $\boxed{1}$

(v) Hence sketch $f(x) = \dfrac{2}{1 + e^{-2x}}$ showing all intercepts with the axes and the equation of any asymptotes. $\boxed{2}$

QUESTION FOURTEEN (15 marks) Start a new answer booklet. **Marks**

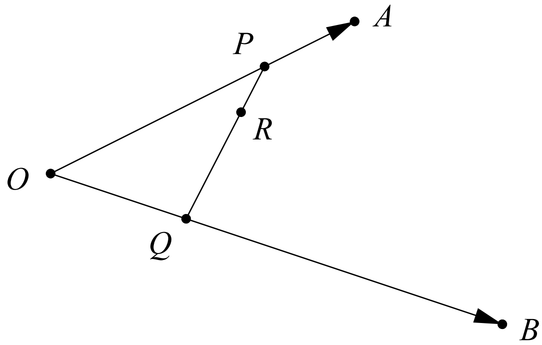
(a) A container is formed by rotating the curve $y = 10\tan^{-1}\dfrac{x}{4}$ around the y-axis from $y = 0$ cm to $y = 12$ cm as shown in the diagram. The container is filled to the top with water.



(i) By noting that $\tan^2\theta = \sec^2\theta - 1$, show that the initial volume V_0 of water in the container is 690 mL, to the nearest mL. $\boxed{3}$

(ii) A hole is made at the bottom of the container and the water drips out a rate of 3 mL/sec. Find the rate of change of the height h of the water when $h = \dfrac{5\pi}{2}$ cm. $\boxed{2}$

(b) In the diagram, $\overrightarrow{OP} = (1-\lambda)\overrightarrow{OA}$, $\overrightarrow{OQ} = \lambda\overrightarrow{OB}$ and $\overrightarrow{PR} = \lambda\overrightarrow{PQ}$, for some real number λ where $0 \le \lambda \le 1$.



(i) Show that $\overrightarrow{OR} = (1-\lambda)^2\overrightarrow{OA} + \lambda^2\overrightarrow{OB}$ where $0 \le \lambda \le 1$. $\boxed{2}$

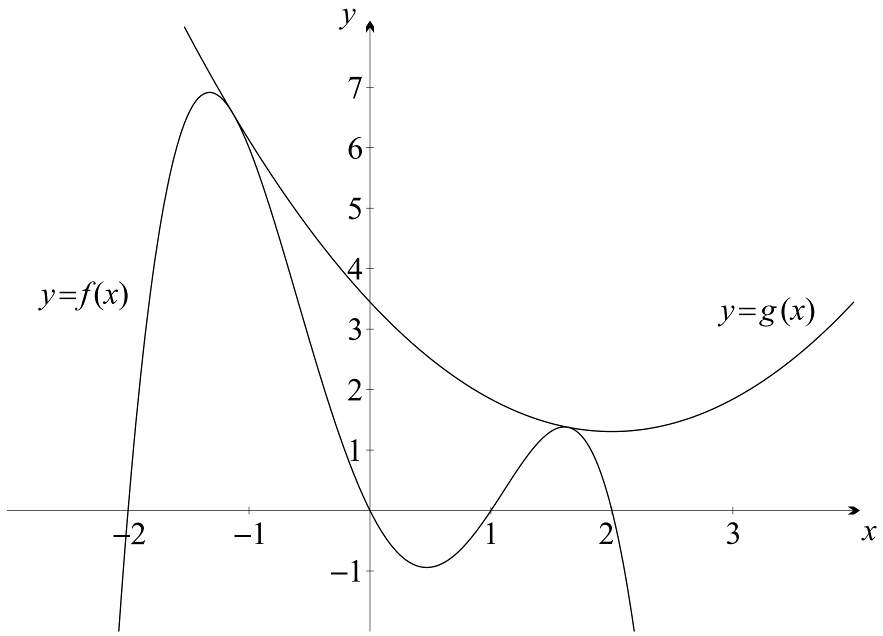
(ii) Show that the Cartesian equation of the curve described by $\overrightarrow{OR}$ when $A = (2, 1)$ and $B = (3, -1)$ is parabolic, and find the coordinates of its vertex. $\boxed{3}$

The question continues on the next page

QUESTION FOURTEEN (Continued)

(c) (i) A monic polynomial $P(x)$ of degree 4 has two zeros m and n, each of which has multiplicity 2. Write down the equation for $P(x)$. $\boxed{1}$

(ii) The curve $f(x) = -x^4 + x^3 + 4x^2 - 4x$ touches the curve $g(x) = a(x-2)^2 + b$ twice as shown in the diagram. $\boxed{4}$



Using the result from part (i), or otherwise, find the values of a and b.

END OF PAPER

EXT 1 2023 TRIAL – SOLUTIONS

1) A

2) $139 = 4 \times 34 + 3$

So at least 35 cars must be the same colour

B.

3) $A \doteq \dfrac{\log_a 2 + \log_a 4}{2} \times 2 = \log_a 2 + \log_a 2^2$

$= 3\log_a 2$ A

4) $f'(2) = -\frac{1}{2}$, $g(2) = 1$

$\left.\dfrac{dy}{dx}\right|_{x=2} = \left.\dfrac{dy}{dt} \cdot \dfrac{dt}{dx}\right|_{x=2}$

$= g'(2) \cdot \dfrac{1}{f'(2)}$

$= 1 \times -2$

$= -2$ D

5) $y = 2\cos\left(\frac{x}{2}\right) \Rightarrow x = 2\cos\left(\frac{y}{2}\right)$

ie $f^{-1}(x) = 2\cos\left(\frac{x}{2}\right)$

Domain $[-2, 2]$ B

6) A

7) ${}^{100}C_{10}\ {}^{100}C_{10}$ B

8) C

9) $g(x) = |f(x) - 2|$ C

10) $f(x) = f\left(\frac{1}{x}\right)$ when $x = 1$ $(x > 0)$ so $a = 1$

$g'(x) = -\frac{1}{x^2} f'\left(\frac{1}{x}\right)$

When $a = 1$, $g'(a) = -f'\left(\frac{1}{a}\right) < 0$

So D is false.

Question 11

a) $P(-1) = (-1)^{17} - 2(-1) - 1 = 0$ ✓

Since $P(-1) = 0$, $x+1$ is a factor of $P(x)$

b) when $x = 3$ & $y = 2$, $\frac{dy}{dx} = 2(1-3) = -4$ ✓

c) MILO has $4! = 24$ arrangements ✓

SAVVAS has $\frac{6!}{2!2!2!} = 90$ arrangements

So Savvas has more arrangements. ✓

d) $\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\,dx$ ✓

$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C$ ✓

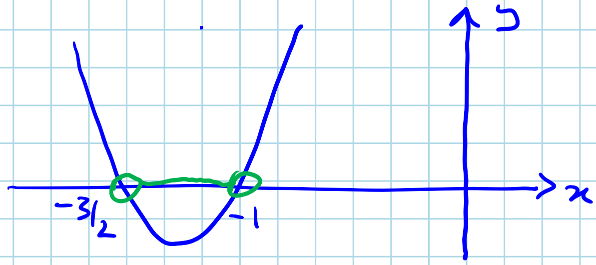
e) $\frac{x}{x+1} > 3$

$x(x+1) > 3(x+1)^2$ ✓

$(x+1)(3(x+1) - x) < 0$

$(x+1)(2x+3) < 0$ ✓

$-\frac{3}{2} < x < -1$ ✓



f) Check $n=1$

$5\times 1^2 - 3\times 1 = 2$ which is div by 2

So true for $n=1$. ✓

Assume true for $n=k$, ie $5k^2 - 3k = 2M$, $M \in \mathbb{Z}^+$

Consider $n=k+1$. RTP $5(k+1)^2 - 3(k+1) = 2N$, $N \in \mathbb{Z}^+$ ✓

$$
\begin{aligned}
LHS &= 5(k^2+2k+1) - 3k - 3 \\
&= 5k^2 + 10k + 5 - 3k - 3 \\
&= (5k^2 - 3k) + 10k + 2 \\
&= 2M + 2(5k+1) \quad \text{from assumption} \\
&= 2N \quad \text{where } N = M + 5k + 1 \\
&= RHS. \quad ✓
\end{aligned}
$$

So by M.I, $5n^2 - 3n$ is div by 2 for $n \in \mathbb{Z}^+$

g) $\underset{\sim}{v} = \begin{pmatrix} -3\sin t \\ 4\cos t \end{pmatrix}$ ✓

$$
\begin{aligned}
|\underset{\sim}{v}| &= \sqrt{9\sin^2 t + 16\cos^2 t} \\
&= \sqrt{9 + 7\cos^2 t} \quad ✓
\end{aligned}
$$

since $0 \le \cos^2 t \le 1$, $\sqrt{9} < |\underset{\sim}{v}| < \sqrt{16}$

ie min speed is 3 m/s & max speed is 4 m/s ✓

(with correct justification)

Question 12

a) $\int_0^1 x(1-x)^7\,dx$ $\qquad u = 1-x$; $x=0 \Rightarrow u=1$

$du = -dx$; $x=1 \Rightarrow u=0$

$= -\int_1^0 (1-u)u^7\,du$ ✓

$= \int_0^1 (u^7 - u^8)\,du$

$= \left.\dfrac{u^8}{8} - \dfrac{u^9}{9}\right|_0^1$ ✓

$= \dfrac{1}{8} - \dfrac{1}{9}$

$= \dfrac{1}{72}$ ✓

b) $\dfrac{dP}{dt} = k(25-P)$

i) $P = 25 - 17e^{-kt}$

LHS $= k(17e^{-kt})$

$= k(25-P)$

$=$ RHS ✓

ii) when $t=1$, $P = 8.072$

$8.072 = 25 - 17e^{-k}$ ✓

$e^{-k} = \dfrac{25-8.072}{17} =$

$k = -\ln\left(\dfrac{25-8.072}{17}\right) = +0.00424$ (3sf) ✓

iii) $20 = 25 - 17e^{-0.00424t}$

$$e^{-0.00424t} = \frac{25-20}{17}$$

$$t = \frac{1}{-0.00424}\ln\left(\frac{25-20}{17}\right)$$

$$= 288.62$$

The pop will be 20 bn in approx 289 years ✓

c) i) $\underset{\sim}{a} + \underset{\sim}{b}$ & $\underset{\sim}{a} - \underset{\sim}{b}$ ✓ $\underset{\sim}{b} - \underset{\sim}{a}$ ok too

ii) $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$

$$= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$$ ✓

$$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$$

$$= 0 \text{ if } |\underset{\sim}{a}| = |\underset{\sim}{b}|$$ ✓

d) i) $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$= \frac{\sin t}{1 - \cos t}$$ ✓

ii) $\frac{\sin t}{1-\cos t} = \frac{1}{\sqrt{3}}$

$$\sqrt{3}\sin t = 1 - \cos t$$

$$\sqrt{3}\sin t + \cos t = 1$$ ✓

$2\sin\left(t + \frac{\pi}{6}\right) = 1$ ✓ by Aux Angle Method

$$t + \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$$

$$t = 0, \frac{2\pi}{3}$$

discard as $1 - \cos 0 = 0$

So $t = \frac{2\pi}{3}$. ✓

Question 13

a) $\left(x-\frac{2}{x}\right)^4 = x^4 - 4x^3\left(\frac{2}{x}\right) + 6x^2\left(\frac{2}{x}\right)^2 - 4x\left(\frac{2}{x}\right)^3 + \left(\frac{2}{x}\right)^4$ ✓

$= x^4 - 8x^2 + 24 - 32x^{-2} + 16x^{-4}$ ✓

b) i) $20t\cos\alpha = 20 \Rightarrow t = \frac{1}{\cos\alpha}$ ✓

$-5t^2 + 20t\sin\alpha = 10$

$-5\sec^2\alpha + 20\tan\alpha = 10$

$-5(1+\tan^2\alpha) + 20\tan\alpha - 10 = 0$

$5\tan^2\alpha - 20\tan\alpha + 15 = 0$

$\tan^2\alpha - 4\tan\alpha + 3 = 0$ ✓

$(\tan\alpha - 3)(\tan\alpha - 1) = 0$

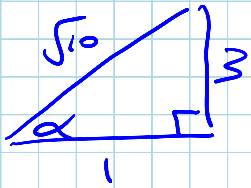
$\tan\alpha = 3$ or $\tan\alpha = 1$

$\alpha = \tan^{-1}(3)$ or $\alpha = 45^\circ$

choose larger angle as $45^\circ < \alpha < 90^\circ$

So $\alpha = \tan^{-1}(3) \doteqdot 72^\circ$ ✓

ii) So $t = \frac{1}{\cos\alpha} = \sqrt{10}$ sec ✓



$-5t^2 + vt = 10$

$-50 + \sqrt{10}\,v = 10$

$v = 6\sqrt{10}$ m/s ✓ (18.97)

c) i) $RHS = \frac{1}{y} + \frac{1}{2-y} = \frac{2-y+y}{y(2-y)}$

$= \frac{2}{y(2-y)}$ ✓

$= LHS$

ii) $\frac{dy}{dx} = y(2-y)$

$\frac{2\,dy}{y(2-y)} = 2\,dx$

$\left(\frac{1}{y} + \frac{1}{2-y}\right)dy = 2\,dx$ ✓

$$\ln|y| - \ln|2-y| = 2x + c$$

$x=0,\ y=1 \Rightarrow \ln 1 - \ln 1 = 0 + c.$ So $c = 0$

$$\ln\left|\frac{y}{2-y}\right| = 2x$$

$$\frac{y}{2-y} = e^{2x} \qquad \left[\text{NB, } \frac{y}{2-y} > 0 \text{ when } y=1\right]$$

$$y = 2e^{2x} - ye^{2x}$$

$$y(1+e^{2x}) = 2e^{2x}$$

$$y = \frac{2e^{2x}}{1+e^{2x}} = \frac{2}{1+e^{-2x}} \ //$$

iii) Let $g(x) = \dfrac{2}{1+e^{-2x}} - 1$

$$= \frac{2-1-e^{-2x}}{1+e^{-2x}}$$

$$= \frac{1-e^{-2x}}{1+e^{-2x}}$$

$$g(-x) = \frac{1-e^{2x}}{1+e^{2x}} \times \frac{e^{-2x}}{e^{-2x}}$$

$$= \frac{e^{-2x}-1}{e^{-2x}+1}$$

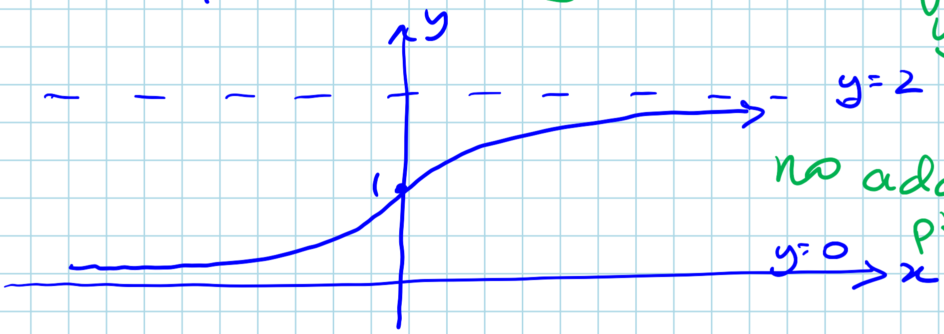
$$= -\left(\frac{1-e^{-2x}}{1+e^{2x}}\right)$$

$$= -g(x)$$

iv) $\left.\begin{aligned} \lim_{x\to\infty} \frac{2}{1+e^{-2x}} &= 2 \\ \lim_{x\to-\infty} \frac{2}{1+e^{-2x}} &= 0 \end{aligned}\right\}$

shape

asymptotes, y-int



no additional pt. necessary.

Question 14

a) i) $V = \pi \int_0^{12} x^2 \, dy$

$x = 4 \tan \frac{y}{10}$

$V = 16\pi \int_0^{12} \tan^2 \frac{y}{10} \, dy$ ✓

$= 16\pi \int_0^{12} \left(\sec^2 \frac{y}{10} - 1\right) dy$

$= 16\pi \left[10 \tan \frac{y}{10} - y \right]_0^{12}$ ✓

$= 16\pi \left[10 \tan 1.2 - 12 \right]$

$\doteqdot 690 \text{ mL}$ ✓

ii) $\frac{dV}{dt} = -3$

$V = 16\pi \int_0^{h} \tan^2 \frac{y}{10} \, dy$

$\frac{dV}{dh} = 16\pi \tan^2 \frac{h}{10}$ ✓ by Fundamental Theorem of Calc.

When $h = \frac{5\pi}{2}$, $\frac{dV}{dh} = 16\pi \tan^2 \frac{\pi}{4} = 16\pi$

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \frac{-3}{16\pi} = -0.06 \text{ cm/min}$ ✓

b) i) $\vec{OR} = \vec{OP} + \vec{PR}$

$= (1-\lambda)\vec{OA} + \lambda \vec{PQ}$

$= (1-\lambda)\vec{OA} + \lambda(\vec{OQ} - \vec{OP})$

$= (1-\lambda)\vec{OA} + \lambda(\lambda \vec{OB} - (1-\lambda)\vec{OA})$ ✓

$= (1-\lambda)\vec{OA} + \lambda^2 \vec{OB} - \lambda(1-\lambda)\vec{OA}$

$= (1-\lambda)\vec{OA}(1-\lambda) + \lambda^2 \vec{OB}$

$= (1-\lambda)^2 \vec{OA} + \lambda^2 \vec{OB}$ ✓

ii) let $\vec{OA} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\vec{OB} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\vec{OR} = \begin{pmatrix} x \\ y \end{pmatrix} = (1-\lambda)^2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda^2 \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$x = 2(1-\lambda)^2 + 3\lambda^2 = 5\lambda^2 - 4\lambda + 2$ (1) ✓

$y = (1-\lambda)^2 - \lambda^2 = 1 - 2\lambda$ (2)

from (2) : $\lambda = \frac{1-y}{2}$

So $x = 5\left(\frac{1-y}{2}\right)^2 - 4\left(\frac{1-y}{2}\right) + 2$ ✓

$= \frac{5}{4}\left(1 - 2y + y^2\right) - 2 + 2y + 2$

$= \frac{5y^2}{4} - \frac{y}{2} + \frac{5}{4}$

$x = \frac{5}{4}\left(y^2 - \frac{2}{5}y\right) + \frac{5}{4}$

$= \frac{5}{4}\left[\left(y - \frac{1}{5}\right)^2 - \frac{1}{25}\right] + \frac{5}{4}$

$= \frac{5}{4}\left(y - \frac{1}{5}\right)^2 + \frac{6}{5}$

Which is parabolic with vertex $\left(\frac{6}{5}, \frac{1}{5}\right)$ ✓

c) i) $P(x) = (x-m)^2(x-n)^2$ ✓

ii) let $f(x) - g(x) = -P(x)$

$-x^4 + x^3 + 4x^2 - 4x - a(x-2)^2 - b = -(x-m)^2(x-n)^2$

$-x^4 + x^3 + 4x^2 - 4x - ax^2 + 4ax - 4a - b$

$= -\left(x^2 - 2xm + m^2\right)\left(x^2 - 2xn + n^2\right)$

$-x^4 + x^3 + (4-a)x^2 + (4a-4)x - 4a - b$ ✓

$= -x^4 + (2m + 2n)x^3 - (n^2 + 4mn + m^2)x^2 + (2mn + 2m^2n)x$

$-m^2n^2$

Equating coefficients:

$2m+2n = 1$ (1) $\rightarrow$ $m+n = \frac{1}{2}$ (5)

$n^2 + 4mn + m^2 = a-4$ (2)

$2mn(m+n) = 4a-4$ (3)

$m^2n^2 = 4a+b$ (4)

✓ at least two correct

sub $m+n=\frac{1}{2}$ into (3) $\rightarrow$ $mn = 4a-4$ (6)

rearrange (2) $\rightarrow (n+m)^2 + 2mn = a-4$ (7)

sub (5) & (6) into (7)

$\left(\frac{1}{2}\right)^2 + 2(4a-4) = a-4$

$7a = 4 - \frac{1}{4} \Rightarrow \boxed{a = \frac{15}{28}}$ ✓

sub $a = \frac{15}{28}$ into (6)

$mn = \frac{15}{7} - 4 = -\frac{13}{7}$

sub $mn = -\frac{13}{7}$, $a = \frac{15}{28}$ into (4)

$\left(-\frac{13}{7}\right)^2 - 4\left(\frac{15}{28}\right) = b$

$\boxed{b = \frac{64}{49}}$ ✓

Alternative soln:

(c)(i) $P(x) = (x-m)^2(x-n)^2$ ✓

(ii) Let $P(x) = g(x) - f(x)$

$= a(x-2)^2 + b + x^4 - x^3 - 4x^2 + 4x$

$= a(x^2-4x+4) + b + x^4 - x^3 - 4x^2 + 4x$

$= x^4 - x^3 + (a-4)x^2 + (4-4a)x + 4a + b$ ✓

$P(x)$ will have 2 zeros of multiplicity 2. Let these zeros be m and n.

$\therefore$ Zeros are m, m, n, n

Sum of zeros: $2m + 2n = 1$

$m + n = \frac{1}{2}$ (1)

Two at a time: $m^2 + mn + mn + mn + mn + n^2 = a - 4$

$m^2 + 4mn + n^2 = a - 4$ (2)

Three at a time: $m^2n + m^2n + mn^2 + mn^2 = -(4 - 4a)$

$2mn(m+n) = 4a - 4$

$mn(m+n) = 2a - 2$ (3)

Product of zeros: $m^2n^2 = 4a + b$ (4)

✓ at least 2 correct

Sub (1) into (3): $mn \times \frac{1}{2} = 2a - 2$

$mn = 4a - 4$ (5)

Sub. (5) into (4): $(4a-4)^2 = 4a + b$

$16a^2 - 32a + 16 = 4a + b$

$b = 16a^2 - 36a + 16$ (6)

In (2): $m^2 + 4mn + n^2 = a - 4$

$(m+n)^2 + 2mn = a - 4$

$(\frac{1}{2})^2 + 2(4a-4) = a - 4$ from (1) and (5)

$\frac{1}{4} + 8a - 8 = a - 4$

$7a = 4 - \frac{1}{4}$

$= \frac{15}{4}$

$a = \frac{15}{28}$ ✓

sub. into (6): $b = 16\left(\frac{15}{28}\right)^2 - 36\left(\frac{15}{28}\right) + 16$

$= \frac{64}{49}$ ✓